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ON Pic($R[X]$) FOR R SEMINORMAL

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0. Introduction

For a commutative ring R with identity, Pic(R) denotes the Picard group of R—that is, the group of projective modules of rank one (under $\otimes_R$). For a large class of rings, including integral domains and Noetherian rings, Pic(R) is isomorphic to the class group Cl(R) of R, which is, by definition, the factor group inv(R)/Prin(R), where inv(R) is the group of invertible fractional ideals of R and Prin(R) is the subgroup of inv(R) consisting of principal fractional ideals. If T is a set of indeterminates over R, then there exists a natural injection $\phi : \text{Pic}(R) \to \text{Pic}(R[T])$. In this paper, we consider the problem of determining conditions under which ϕ is an isomorphism. In [5, Theorem 3.6], Traverso restricted his attention to Noetherian reduced rings with finite integral closure and solved the problem in that case. There, ϕ is an isomorphism if and only if R is seminormal (several characterizations of which shall be offered in Theorem 1.1). In Section 1, we extend Traverso's Theorem 3.6 to an arbitrary integral domain and an arbitrary set of indeterminates; this is Theorem 1.6. Following the proof of Theorem 1.6, we note that the theorem remains true if the hypothesis that R is an integral domain is replaced by the assumption that R is a Noetherian reduced ring – that is, Traverso's theorem is true without the hypothesis that R has finite integral closure. Also, the assumption of reduced rings does not actually serve as a limitation on any of the results because Pic(R) is canonically isomorphic to Pic(R_{red}), where R_{red} is R modulo its nilradical. In Section 2 we offer an example that demonstrates the necessity of the stronger hypothesis of domain in the non-Noetherian case. This is an example of a reduced ring R which is its own total quotient ring (hence R is seminormal), but for which ϕ is not an isomorphism.

Throughout the paper we use ′ to denote integral closure. Thus, the integral closure of a ring Y is denoted by Y'.

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1. Seminormality and the domain case

Traverso in [5] defines a ring R to be *seminormal* if

$$R = \{x \in R' \mid \text{for each } P \in \operatorname{Spec}(R),\ \bar{x} \in R_P + J(R'_P)\},$$

where $J(R'_P)$ is the Jacobson radical of $(R')_{R-P}$. We begin by establishing two equivalent forms of Traverso's definition that are frequently more tractable in applications.

Theorem 1.1. *Let R be a ring. The following conditions are equivalent.*
(1) *R is seminormal.*
(2) *For each $y \in R'$, the conductor of R in $R[y]$ is a radical ideal of $R[y]$.*
(3) *R contains each element y of R' such that $y^n \in R$ for all sufficiently large n.*

Proof. Lemma 1.3 of [5] shows that (1) implies (2), and the equivalence of (2) and (3) is straightforward.

(2) → (1). This is the essential content of Lemma 1.7 of [5]; that result is stated for Noetherian rings, but the Noetherian hypothesis isn't necessary for the proof. Because the statement of Traverso's result differs widely from ours, we choose for the sake of clarity to present here a suitable modification of Traverso's proof.

In the notation and terminology of [5], the subring

$$\{y \in R' \mid \bar{y} \in R_P + J(R'_P) \text{ for all } P \in \operatorname{Spec}(R)\}$$

of R' is denoted by ${}^+R$ and is called the *seminormal closure* of R. Traverso shows [5, (1.1)] that ${}^+R$ is the largest subring of R' containing R such that the following two conditions are satisfied:

(i) for each prime P of R, there exists a unique prime ${}^+P$ or ${}^+R$ lying over P, and

(ii) the canonical injection of R_P/PR_P into ${}^+R_{{}^+P}/{}^+P\,{}^+R_{{}^+P}$ is surjective for each prime P of R.

It follows that (i) and (ii) are satisfied for the extension $R \subseteq R[y]$ for each $y \in {}^+R$.

Assume then that (2) holds and that R is not seminormal. Choose $y \in {}^+R - R$, let C be the conductor of R in $R[y]$, and let Q be a minimal prime of C in R. Then $\bar{y} \in R'_Q$, $\bar{y} \notin R_Q$, and $\operatorname{Rad}_{R_Q}(CR_Q) = QR_Q$. It is straightforward to show that condition (2) is satisfied for the ring R_Q; moreover

$$\bar{y} \in \bigcap \{(R_Q)_{PR_Q} + J((R'_Q)_{PR_Q}) \mid PR_Q \in \operatorname{Spec}(R_Q)\}.$$

It follows that there is a unique prime ideal of $R_{Q[\bar{y}]}$ lying over QR_Q in R_Q, and this prime ideal is $\operatorname{Rad}_{R_Q[\bar{y}]}(QR_Q[\bar{y}])$, which is QR_Q since QR_Q is the conductor of R_Q in $R_Q[\bar{y}]$, a radical ideal of $R_Q[\bar{y}]$. Also, the canonical injection of R_Q/QR_Q into $R_Q[\bar{y}]/QR_Q$ is an isomorphism. This implies, however, that $R_Q = R_Q[\bar{y}]$, a contradiction. This completes the proof of Theorem 1.1.

Lemma 1.2. *Assume that D is a seminormal domain and A is a subring of D such that $A = A' \cap D$. Then A is seminormal.*

Proof. Using condition (3) of Theorem 1.1, the verification is routine.

We remark that the inclusion $A' \subseteq D'$ is crucial in the proof of Lemma 1.2. Thus Lemma 1.2 fails if D is merely a reduced seminormal ring, and consequently, we cannot prove (mercifully, it being false) that the injection from Pic(D) into Pic($D[T]$) is surjective in the more general case.

In order to prove our main result, Theorem 1.6, we need to establish a relationship between direct limits and Picard groups. While this connection is well understood by specialists, it does not seem to appear in the literature and we produce a proof of it here.

Theorem 1.3. *Let* $\{R_\alpha, f_{\alpha\beta}\}$ *be a direct system of rings and let* $R = \underrightarrow{\lim}\{R_\alpha\}$. *Then* $\mathrm{Pic}(R) = \underrightarrow{\lim}\{\mathrm{Pic}(R_\alpha)\}$.

Before starting on the proof, we develop a matrix characterization of rank-one projective modules. If S is a ring and A is an $n \times n$ matrix with entries in S, then we may regard the rows of A as elements of the free module $S^{(n)}$ and associate with A the submodule of $S^{(n)}$ – call it $M(A)$ – generated by the rows.

Next we say that a matrix A satisfies (*) provided

(i) the ideal generated by its entries is S, and

(ii) all of its 2×2 minors are zero.

Lemma 1.4. (a) *If A satisfies* (*), *then $M(A)$ is a rank-one projective summand of* $S^{(n)}$.

(b) *If M is a rank-one projective S-module with n generators, then there exists an $n \times n$ matrix A satisfying* (*) *with* $M \simeq M(A)$.

Proof. A preliminary remark on notation is in order. If $s_1, \ldots, s_n \in S$, then throughout the proof we use $(s_1, \ldots, s_n)$ or $(\{s_i\})$ to denote the ideal of S generated by $\{s_i\}_{i=1}^n$ and we use $\langle s_1, \ldots, s_n \rangle$ to denote the row vector in $S^{(n)}$ with ith coordinate s_i for each i.

(a) If A satisfies (*), then it also satisfies (*) when considered naturally as a matrix over a localization of S. If $M(A)$ is locally a rank-one projective summand, then $S^{(n)}/M(A)$ is locally a rank-$(n-1)$ projective, and hence globally as well [1, p. 112, II.5.3.2]. Hence, without loss of generality, we may assume that S is quasi-local.

Since $(\{a_{ij}\}) = S$, it follows that some a_{ij} is a unit. By symmetry, we can assume that a_{11} is a unit. Clearly, $\langle a_{11}, \ldots, a_{1n} \rangle S$ is a rank-one free summand of $S^{(n)}$. To complete the proof of (a), it suffices to show that for each i, $\langle a_{i1}, \ldots, a_{in} \rangle = a_{11}^{-1} a_{i1} \cdot \langle a_{11}, \ldots, a_{1n} \rangle$ – that is, $a_{11} a_{ij} = a_{1j} a_{i1}$ for all i, j. The vanishing of the 2×2 minors gives this.

(b) Since M has n generators, there exists a (necessarily split) epimorphism $S^{(n)} \to M$. The splitting map gives an isomorphism between M and its image, and so we may assume that M is a direct summand of $S^{(n)}$. Regarding some n-element generating set for M as the rows of a matrix, we obtain $M = M(A)$ for some matrix A.

We prove that A satisfies (*). To do so, it is enough to show that A satisfies (*) at all localizations, and hence we may assume that S is quasi-local with maximal ideal P. Noting that $S^{(n)}/M(A)$ is free of rank $(n-1)$ and $S^{(n)}/PS^{(n)}$ requires n generators, we see that $M(A) \not\subseteq PS^{(n)}$ – that is, some entry of A is a unit. If a_{ij} is a unit, then $\langle a_{i1}, \ldots, a_{in}\rangle S$ is a rank-one direct summand of $S^{(n)}$. Inasmuch as proper containment of rank-one summands is impossible, $M(A) = \langle a_{i1}, \ldots, a_{in}\rangle S$. Thus, each row is a multiple of the ith row and the vanishing of the 2×2 minors follows immediately.

Proof of Theorem 1.3. There is an obvious canonical homomorphism

$$\bar{f}: \varinjlim\{\mathrm{Pic}(R_\alpha)\} \to \mathrm{Pic}(R),$$

and what we mean by equality in Theorem 1.3 is that this natural map is an isomorphism. First we show that $\bar{f}$ is epic. Suppose $[M] \in \mathrm{Pic}(R)$. Employing Lemma 1.4(b), assume that $M = M(A)$ with A satisfying (*). Since $(\{a_{ij}\}) = R$, there exists $\{b_{ij}\}$ with $\sum\sum a_{ij}b_{ij} = 1$. Since R is a direct limit, we can find $R_\beta \in \{R_\alpha\}$ such that R_β contains a set of elements $\{\tilde{a}_{ij}, \tilde{b}_{ij}\}$ with $f(\tilde{a}_{ij}) = a_{ij}$ and $f(\tilde{b}_{ij}) = b_{ij}$. Form the matrix $\tilde{A} = (\tilde{a}_{ij})$ and let $\{c_k\}$ be the set of 2×2 minors of $\tilde{A}$. Note that

$$\{1 - \textstyle\sum\sum \tilde{a}_{ij}\tilde{b}_{ij}\} \cup \{c_k\} \subseteq \operatorname{Ker} f$$

and so there exists $R_\delta \in \{R_\alpha\}$ and $f_{\beta\delta}: R_\beta \to R_\delta$ with this set in the kernel of $f_{\beta\delta}$. Letting $\overset{\approx}{A} = f_{\beta\delta}(\tilde{A})$, we observe that $\overset{\approx}{A}$ satisfies (*) and so $[M(\overset{\approx}{A})] \in \mathrm{Pic}(R_\delta)$.

Next, since $M(\overset{\approx}{A})$ is a direct summand of $R_\delta^{(n)}$, $(0) \to M(\overset{\approx}{A}) \otimes R \to R_\delta^{(n)} \otimes R$ is exact, and so $M(A) \otimes R \approx M(A) = M$. Thus $\bar{f}$ is epic.

Now assume $[M_1], [M_2] \in \varinjlim\{\mathrm{Pic}(R_\alpha)\}$ and $M_1 \otimes R \simeq M_2 \otimes R$. We can assume that M_1 and M_2 are modules over the same R_β and for some n, there exist $n\times n$ matrices A_1, A_2 with entries in R_β such that $M_i = M(A_i)$. As noted above, $M(A_i) \otimes R \simeq M(f(A_i))$ and so $M(f(A_1)) \simeq M(f(A_2))$. Since $M(f(A_1))$ is a direct summand of $R^{(n)}$, this isomorphism can be extended to an endomorphism of $R^{(n)}$ – that is, to a matrix map. Thus, there exists a matrix B with $M(f(A_1)B) = A(f(A_2))$.

It is easy to see that $M(C_1) \subseteq M(C_2)$ if and only if there exists a matrix D with $C_1 = DC_2$. Thus, there exist D_1, D_2 with $f(A_1)B = D_2 f(A_2)$ and $D_1 f(A_1)B = f(A_2)$. Next select $R_\delta \in \{R_\alpha\}$ and $f_{\beta\delta}: R_\beta \to R_\delta$ such that R_δ contains preimages of each entry of the matrices B, D_1, D_2. We denote $(\tilde{b}_{ij})$ by $\tilde{B}$ and likewise define $\tilde{D}_1, \tilde{D}_2$. Set $\tilde{A}_i = f_{\beta\delta}(A_i)$. Then $\tilde{A}_1\tilde{B} - \tilde{D}_1\tilde{A}_2$ and $\tilde{D}_1\tilde{A}_1\tilde{B} - \tilde{A}_2$ are in $\operatorname{Ker} f^*$, where $f^*: \mathrm{Mat}(R_\delta) \to \mathrm{Mat}(R)$. This means we can find $f_{\delta\sigma}: R_\delta \to R_\sigma$ with

$$f^*_{\delta\sigma}(\tilde{A}_1\tilde{B} - \tilde{D}_2\tilde{A}_2) = f^*_{\delta\sigma}(\tilde{D}_1\tilde{A}_1\tilde{B} - \tilde{A}_2) = 0.$$

Using $\approx$ to denote matrices with entries in R_σ, this yields $M(\overset{\approx}{A}_1\overset{\approx}{B}) = M(\overset{\approx}{A}_2)$. If, however, $\theta: R_\sigma^{(n)} \to R_\sigma^{(n)}$ is multiplication on the right by $\overset{\approx}{B}$, this gives $\theta(M(\overset{\approx}{A}_1)) = M(\overset{\approx}{A}_2)$. Since an epimorphism of rank-one projectives splits, and so is an isomorphism, $M(\overset{\approx}{A}_1) \simeq M(\overset{\approx}{A}_2)$; that is, $M(A_1) \otimes R_\sigma \simeq M(A_2) \otimes R_\sigma$ and so

$$[M_1] = [M_2] \in \varinjlim\{\mathrm{Pic}(R_\alpha)\}.$$

Theorem 1.5. *Assume that R is a reduced ring and T is a set of indeterminates over R. If R is not seminormal, then the canonical injection of* Pic(R) *into* Pic($R[T]$) *is not surjective.*

Proof. Let π be the prime subring of R, and consider the family $\{S_\alpha\}$ of all finitely generated ring extensions of π that are subrings of R. Each S_α is a Noetherian reduced ring and R is the direct union of the family $\{S_\alpha\}$. Moreover, if $\{T_\beta\}$ is the family of finite subsets of T, then $\{S_\alpha[T_\beta]\}$ is a directed family of Noetherian subrings of $R[T]$ and $R[T]$ is the direct union of this family. Considering the diagram

$$\begin{array}{ccccc} \mathrm{Pic}(R_\alpha) & \longrightarrow & \mathrm{Pic}(R_\beta) & \longrightarrow & \mathrm{Pic}(R) \\ \Big\downarrow{\scriptstyle \phi_\alpha} & & \Big\downarrow{\scriptstyle \phi_\beta} & & \Big\downarrow{\scriptstyle \phi} \\ \mathrm{Pic}(R_\alpha[T_\alpha]) & \longrightarrow & \mathrm{Pic}(R_\beta[T_\beta]) & \longrightarrow & \mathrm{Pic}(R[T]), \end{array}$$

where $R_\alpha \subseteq R_\beta$, $T_\alpha \subseteq T_\beta$, and where ϕ_α, ϕ_β, and ϕ are the canonical maps, we can show that ϕ is not surjective by finding an element of Pic($R_\alpha[T_\alpha]$) – call it F – such that $F \otimes R_\beta[T_\beta] \notin \mathrm{Im}(\phi_\beta)$ for each R_β containing R_α.

Since R_α and R_β are Noetherian, we employ the identification $\mathrm{Pic}(R_\sigma) = \mathrm{Cl}(R_\sigma)$. Pick $t \in T$. Since R is not seminormal, there exists an element $y \in R'$, $y \notin R$ such that $y^n \in R$ for $n \geq 2$. Then $y = c/d$, where d is not a zero divisor in R. Choose α so that $y^2, y^3, c, d \in R_\alpha$, and let T_α be a finite subset of T containing t. Let $F_\alpha = (1 - yt, y^2)$ be the fractional ideal of $R_\alpha[T_\alpha]$ generated by $1 - yt$ and y^2. We have

$$(1-yt, y^2)(1+yt, y^2) = (1-y^2t^4, y^2-y^3t, y^2+y^3t, y^4) = R_\alpha[T_\alpha],$$

so F_α is invertible. Assume that the image of some σ_β contains $F_\alpha \otimes R_\beta[T_\beta] = F_\alpha R_\beta[T_\beta] = F_\beta$. Then $F_\beta = AfR_\beta[T_\beta]$ for some invertible fractional ideal A of R_β and some element f in the total quotient ring of $R_\beta[T_\beta]$. If S is the total quotient ring of R_β, then we have

$$S[T_\beta] = F_\beta S[T_\beta] = AS[T_\beta] \cdot fS[T_\beta] = fS[T_\beta].$$

Thus f is a unit of $S[T_\beta]$, and since S is reduced, it follows that f is a unit of S [4, p. 683]. Therefore $F_\beta = BR_\beta[T_\beta]$, where $B = Af$ is an invertible fractional ideal of R_β. Now B is necessarily the fractional ideal of R_β generated by the coefficients of a set of generators of F_β. That is, B is the fractional ideal of R_β generated by the set $\{1, y, y^2\}$. Consequently, $1 \in F_\beta = (y^2, 1 - yt)$. But then

$$1 \cdot (1+yt) \in (y^2, 1-yt)(y^2, 1+yt) = R_\beta[T_\beta],$$

and so $y \in R_\beta \subseteq R$ – a contradiction. Therefore, $F_\beta \notin \mathrm{Im}(\phi_\beta)$ and the proof of Theorem 1.5 is complete.

Theorem 1.5 is the last preliminary result needed for the proof of the main theorem.

Theorem 1.6. *Let D be an integral domain and let T be a set of indeterminates over D. The natural monomorphism* $\mathrm{Pic}(D) \to \mathrm{Pic}(D[T])$ *is surjective if and only if D is seminormal.*

Proof. In view of Theorem 1.5, we can establish Theorem 1.6 by showing that if D is a seminormal integral domain, then $\mathrm{Pic}(D) \to \mathrm{Pic}(D[T])$ is surjective. Let R be the prime subring of D. As in the proof of Theorem 1.5, we think of D as the direct union of its family $\{S_\alpha\}$ of subrings that are finitely generated as algebras over R. Since R is either Z or a prime field, each S_α is a Noetherian domain with finite integral closure [3, p. 133]. Thus $D_\alpha = (S_\alpha)' \cap D$ is again a finitely generated algebra with finite integral closure. By (1.2), D_α is seminormal. Clearly, $D = \underrightarrow{\lim}\{D_\alpha\}$ and $D[T] = \underrightarrow{\lim}\{D_\alpha[T_\beta]\}$, where T_β is a finite subset of T. By Traverso's Theorem 3.6, $\mathrm{Pic}(D_\alpha) \to \mathrm{Pic}(D_\alpha[T_\beta])$ is epic. As a direct limit of a family of epimorphisms is epic, we obtain an epimorphism $\underrightarrow{\lim}\{\mathrm{Pic}(D_\alpha)\} \to \underrightarrow{\lim}\{\mathrm{Pic}(D_\alpha[T_\beta])\}$. Applying Theorem 1.4, this completes the proof of Theorem 1.6.

We remark that the "only if" part of Theorem 1.6 has been obtained independently by Brewer and Costa [2].

Theorem 1.6 remains valid if the hypothesis that D is an integral domain is replaced by the condition that D is a reduced Noetherian ring – that is, the conclusion of Traverso's Theorem 3.6 does not require the hypothesis that the ring has finite integral closure. The proof in this case can be obtained by an appropriate modification of the proof of Theorem 1.6. Essentially two problems arise in carrying the proof of Theorem 1.6 through. One is the statement that a ring S of the form $R[\{$a finite subset of $D\}]$ has finite integral closure; this can be obtained from the domain case and from the fact that the total quotient ring of S, a Noetherian reduced ring, is a finite direct sum of fields. The second problem arises from the statement that $D_\alpha = (S_\alpha)' \cap D$ is seminormal. This is Lemma 1.2, which depends upon the fact that $(S_\alpha)' \subseteq D'$. This problem can be handled by placing a further restriction on the subrings S_α. Thus, since D is Noetherian, there are finitely many associated primes $P_1, P_2, \ldots, P_s$ of (0) in D. For each i, there exists $x_i \in D$ such that $P_i = \mathrm{Ann}_D(x_i)$. Note that each subring A of D containing $\{x_i\}_{i=1}^s$ has the property that

$$\{\text{zero divisors of } D\} \cap A = (\textstyle\bigcup P_i) \cap A = \{\text{zero divisors of } A\},$$

and hence regular elements of A are regular in D so that $A' \subseteq D'$. In the notation of Theorem 1.6, we only consider those subrings S_α which contain $\{x_i\}_{i=1}^s$. This solves the problem and is harmless because D is still the direct union of the remaining subrings D_α.

2. An example

As stated in the introduction, we give in this section an example of a reduced total quotient ring R such that $\text{Pic}(R) \to \text{Pic}(R[T])$ is not surjective. Thus, it is impossible to extend Theorem 1.6 to arbitrary reduced rings.

Example 2.1. Let K be a field, let $X \cup \{Y_i\}_{i=1}^{\infty}$ be a set of indeterminates over K, and let M be the ideal of $K[X^2, X^3, Y_1, Y_2, \ldots]$ generated by the set $\{X^2, X^3, Y_1, Y_2, \ldots\}$. Define R to be the ring

$$R = (K[X^2, X^3, Y_1, Y_2, \ldots])_M / (\{X^2 Y_i, X^3 Y_i, Y_i Y_j \mid i \neq j\})_M.$$

Let x, y_i represent the images of X, Y_i, respectively, in R, and set

$$R_n = (K[x^2, x^3, y_1, \ldots, y_n])_{M_n},$$

where M_n is the ideal of $K[x^2, x^3, y_1, \ldots, y_n]$ generated by the set $\{x^2, x^3, y_1, \ldots, y_n\}$. Then $R = \bigcup_{n=1}^{\infty} R_n$.

First note that $z_n = x^2 + \sum_{i=1}^{n} y_i$ is not a zero divisor in R_n. Hence $x = xz_n/z_n \in R_n'$. So while R is a seminormal ring (being, in fact, its own total quotient ring), the rings R_n are not seminormal.

Consider the fractional ideal $(1 - xt, x^2)R_n[T]$. In the proof of Theorem 1.5, we showed that this was an element of $\text{Pic}(R_n[T])$ that was not in $\text{Im}\, \phi_n$, where $\phi_n : \text{Pic}(R_n) \to \text{Pic}(R_n[T])$. Since

$$(1 - xt, x^2)R_n[T] \otimes R_m[T] \simeq (1 - xt, x^2)R_m[T]$$

whenever $m > n$, we have (again as in the proof of Theorem 1.5),

$$(1 - xt, x^2)R_n[T] \otimes R[T] \notin \phi(\text{Pic}(R)).$$

Thus R is the desired example.

It seems worth remarking that

$$(1 - xt, x^2)R_n[T] \otimes R[T] \simeq (1 - x^2t^2, x^2 + x^3t)R[T] \in \text{Cl}(R[T]).$$

Hence $\text{Cl}(R) \to \text{Cl}(R[T])$ is not surjective either.

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